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ON THE CONVERGENCE OF SPATIAL HOMEOMORPHISMS

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Various theorems on the convergence of general spatial homeomorphisms are proved and, on this basis, convergence theorems for classes of the so-called ring Q -homeomorphisms are obtained. These results will have wide applications to Sobolev's mappings.

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Доказаны различные теоремы о сходимости общих пространственных гомеоморфизмов и, на этой основе, получены теоремы о сходимости для так называемых кольцевых Q -гомеоморфизмов. Эти результаты будут иметь широкие приложения к отображениям классов Соболева.

1. Introduction. We give here foundations of the convergence theory for general homeomorphisms in the space and then develop the convergence theory for the so-called Q -homeomorphisms. The ring Q -homeomorphisms have been introduced first in a plane in connection with the study of the degenerate Beltrami equations, see e.g. the papers [22]–[26] and the monographs [8] and [16]. The theory of ring Q -homeomorphisms is applicable to various classes of mappings with finite distortion intensively investigated in many recent works, see e.g. [13] and [16] and further references therein. The present paper is a natural continuation of our previous works [20] and [21].

Given a family Γ of paths γ in \mathbb{R}^n , $n \geq 2$, a Borel function $\rho: \mathbb{R}^n \rightarrow [0, \infty]$ is called *admissible* for Γ , abbr. $\rho \in \text{amd } \Gamma$, if $\int_{\gamma} \rho(x) |dx| \geq 1$ for each $\gamma \in \Gamma$. The *modulus* of Γ is the quantity

$$M(\Gamma) = \inf \left\{ \int_{\mathbb{R}^n} \rho^n(x) dm(x) : \rho \in \text{amd } \Gamma \right\}.$$

Given a domain D and two subsets E and F of $\overline{\mathbb{R}^n}$, $n \geq 2$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma: [a, b] \rightarrow \overline{\mathbb{R}^n}$ which join E and F in D , i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E, F) = \Gamma(E, F, \overline{\mathbb{R}^n})$ if $D = \overline{\mathbb{R}^n}$. A *ring domain*, or shortly a *ring* in $\overline{\mathbb{R}^n}$, is a domain R in $\overline{\mathbb{R}^n}$ whose complement has two connected components. Let R be a ring in $\overline{\mathbb{R}^n}$. If C_1 and C_2 are the connected components of $\overline{\mathbb{R}^n} \setminus R$, we write $R = R(C_1, C_2)$. The *capacity* of R can be defined by the equality $\text{cap } R(C_1, C_2) = M(\Gamma(C_1, C_2, R))$, see e.g.

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5.49 in [30]. Note also that $M(\Gamma(C_1, C_2, R)) = M(\Gamma(C_1, C_2))$, see e.g. Theorem 11.3 in [29]. A *conformal modulus* of a ring $R(C_1, C_2)$ is defined by

$$\text{mod } R(C_1, C_2) = \left(\frac{\omega_{n-1}}{M(\Gamma(C_1, C_2))} \right)^{1/(n-1)},$$

where ω_{n-1} denotes the area of the unit sphere in \mathbb{R}^n , see e.g. (5.50) in [30].

The following notion was motivated by the ring definition of quasiconformality in [7]. Let D be a domain in \mathbb{R}^n , $Q: D \rightarrow (0, \infty)$ be a (Lebesgue) measurable function. Set

$$A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n: r_1 < |x - x_0| < r_2\}, S(x_0, r_i) = \{x \in \mathbb{R}^n: |x - x_0| = r_i\} \ (i \in \{1, 2\}).$$

We say (see [20]) for the spatial case, that a homeomorphism f of D into $\overline{\mathbb{R}^n}$ is a *ring Q -homeomorphism at a point $x_0 \in D$* if

$$M(\Gamma(f(S_1), f(S_2))) \leq \int_A Q(x) \cdot \eta^n(|x - x_0|) dm(x) \quad (1)$$

for every ring $A = A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < r_0 = \text{dist}(x_0, \partial D)$, $S_i = S(x_0, r_i)$, $i \in \{1, 2\}$, and for every Lebesgue measurable function $\eta: (r_1, r_2) \rightarrow [0, \infty]$ such that $\int_{r_1}^{r_2} \eta(r) dr \geq 1$.

If condition (1) holds at every point $x_0 \in D$, then we also say that f is a *ring Q -homeomorphism in the domain D* .

2. On BMO and FMO functions. Recall that a real valued function $\varphi \in L^1_{\text{loc}}(D)$, given in a domain $D \subset \mathbb{R}^n$, is said to be of *bounded mean oscillation* by John and Nirenberg, abbr. $\varphi \in \text{BMO}(D)$ or simply $\varphi \in \text{BMO}$, see [10], if

$$\|\varphi\|_* = \sup_{B \subset D} \frac{1}{|B|} \int_B |\varphi(x) - \varphi_B| dm(x) < \infty,$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \frac{1}{|B|} \int_B \varphi(x) dm(x)$$

is the average of the function φ over B . For connections of BMO functions with quasiconformal and quasiregular mappings, see e.g. [1], [2], [11], [17] and [19].

Following [9], we say that a function $\varphi: D \rightarrow \mathbb{R}$ has *finite mean oscillation* at a point $x_0 \in D$ if

$$\overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0, \varepsilon)} |\varphi(x) - \tilde{\varphi}_\varepsilon| dm(x) < \infty, \quad (2)$$

where $\tilde{\varphi}_\varepsilon = \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0, \varepsilon)} \varphi(x) dm(x)$ is the average of the function $\varphi(x)$ over the ball $B(x_0, \varepsilon) = \{x \in \mathbb{R}^n: |x - x_0| < \varepsilon\}$. Note that under (2) it is possible that $\tilde{\varphi}_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

We also say that a function $\varphi: D \rightarrow \mathbb{R}$ is of *finite mean oscillation in the domain D* , abbr. $\varphi \in \text{FMO}(D)$ or simply $\varphi \in \text{FMO}$, if φ has finite mean oscillation at every point $x \in D$. Note that FMO is not BMO_{loc} , see examples in [16], p. 211. It is well-known that $L^\infty(D) \subset \text{BMO}(D) \subset L^p_{\text{loc}}(D)$ for all $1 \leq p < \infty$, see e.g. [10] and [19], but $\text{FMO}(D) \not\subset L^p_{\text{loc}}(D)$ for any $p > 1$.

Recall some facts on finite mean oscillation from [9], see also 6.2 in [16].

Proposition 1. *If, for some numbers $\varphi_\varepsilon \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0, \varepsilon)} |\varphi(x) - \varphi_\varepsilon| dm(x) < \infty,$$

then φ has finite mean oscillation at x_0 .

Corollary 1. *If, for a point $x_0 \in D$,*

$$\overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0, \varepsilon)} |\varphi(x)| dm(x) < \infty,$$

then φ has finite mean oscillation at x_0 .

Lemma 1. *Let $\varphi: D \rightarrow \mathbb{R}, n \geq 2$, be a nonnegative function with a finite mean oscillation at $0 \in D$. Then*

$$\int_{\varepsilon < |x| < \varepsilon_0} \frac{\varphi(x) dm(x)}{(|x| \log \frac{1}{|x|})^n} = O\left(\log \log \frac{1}{\varepsilon}\right)$$

as $\varepsilon \rightarrow 0$ for a positive $\varepsilon_0 \leq \text{dist}(0, \partial D)$.

This lemma takes an important part in many applications to the mapping theory as well as to the theory of the Beltrami equations, see e.g. the monographs [8] and [16].

3. Convergence of general homeomorphisms. In what follows, we use in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$ where π is the stereographic projection of $\overline{\mathbb{R}}^n$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in \mathbb{R}^{n+1} , i.e.

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x \neq \infty, \quad y \neq \infty, \quad h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}.$$

It is clear that $\overline{\mathbb{R}}^n$ is homeomorphic to the unit sphere S^n in \mathbb{R}^{n+1} .

The *spherical (chordal) diameter* of a set $E \subset \overline{\mathbb{R}}^n$ is $h(E) = \sup\{h(x, y) : x, y \in E\}$. We also define $h(z, E)$ for $z \in \overline{\mathbb{R}}^n$ and $E \subseteq \overline{\mathbb{R}}^n$ as a infimum of $h(z, y)$ over all $y \in E$ and $h(F, E)$ for $F \subseteq \overline{\mathbb{R}}^n$ and $E \subseteq \overline{\mathbb{R}}^n$ as the infimum of $h(z, y)$ over all $z \in F$ and $y \in E$. Later on, we also use the notation $B^*(x_0, \rho)$, $x_0 \in \overline{\mathbb{R}}^n$, $\rho \in (0, 1)$, for the balls $\{x \in \overline{\mathbb{R}}^n : h(x, x_0) < \rho\}$ with respect to the spherical metric.

Let us start with a simple consequence of the well-known Brouwer theorem on invariance of domains.

Corollary 2. *Let U be an open set in $\overline{\mathbb{R}}^n$ and let $f: U \rightarrow \overline{\mathbb{R}}^n$ be continuous and injective. Then f is a homeomorphism of U onto $V = f(U)$.*

Proof. Let $y_0 \in f(D)$ and $x_0 := f^{-1}(y_0)$. Set $B = B^*(x_0, \varepsilon_0)$ where $0 < \varepsilon_0 < h(x_0, \partial D)$. Then $\overline{B} \subset D$. Note that the mapping $f_0 := f|_{\overline{B}}$ is injective and continuous and maps the compactum \overline{B} into the Hausdorff topological space \mathbb{R}^n . Consequently, f_0 is a homeomorphism of \overline{B} onto the topological space $f_0(\overline{B})$ with the topology induced by that of \mathbb{R}^n (see Theorem 41.III.3 in [15]). By the Brouwer theorem on invariance domains (see e.g. Theorem 4.7.16 in [28]), f maps the ball B onto a domain in \mathbb{R}^n as a homeomorphism. Hence the mapping $f^{-1}(y)$ is continuous at the point y_0 . Thus, $f: D \rightarrow \overline{\mathbb{R}}^n$ is a homeomorphism. \square

The *kernel* of a sequence of open sets $\Omega_l \subset \overline{\mathbb{R}}^n$, $l = 1, 2, \dots$ is the open set

$$\Omega_0 = \text{Kern } \Omega_l := \bigcup_{m=1}^{\infty} \text{Int} \left(\bigcap_{l=m}^{\infty} \Omega_l \right),$$

where $\text{Int } A$ denotes the set consisting of all inner points of A ; in other words, $\text{Int } A$ is the union of all open balls in A with respect to the spherical distance.

The following statement for the plane case can be found in [3], see also Proposition 2.7 in [8].

Proposition 2. *Let $g_l: D \rightarrow D'_l$, $D'_l := g_l(D)$, be a sequence of homeomorphisms defined on a domain $D \subset \overline{\mathbb{R}}^n$. Suppose that g_l converges as $l \rightarrow \infty$ locally uniformly with respect to the spherical (chordal) metric to a mapping $g: D \rightarrow D' := g(D) \subset \overline{\mathbb{R}}^n$ which is injective. Then g is a homeomorphism and $D' \subset \text{Kern } D'_l$.*

Proof. First of all, the mapping g is continuous as a locally uniform limit of continuous mappings, see e.g. Theorem 13.VI.3 in [14]. Thus, by Corollary 2 g is a homeomorphism.

Now, let y_0 be a point in D' . Consider the spherical ball $B^*(z_0, \rho)$ where $z_0 := g^{-1}(y_0) \in D$ and $\rho < h(z_0, \partial D)$. Then $r_0 := \min_{z \in \partial B^*(z_0, \rho)} h(y_0, g(z)) > 0$. There is an integer N large enough such that $g_l(z_0) \in B^*(y_0, r_0/2)$ for all $l \geq N$ and simultaneously

$$B^*(y_0, r_0/2) \cap g_l(\partial B^*(z_0, \rho)) = B^*(y_0, r_0/2) \cap \partial g_l(B^*(z_0, \rho)) = \emptyset$$

because $g_l \rightarrow g$ ($l \rightarrow +\infty$) uniformly on the compact set $\partial B^*(z_0, \rho)$. Hence by the connectedness of balls

$$B^*(y_0, r_0/2) \subset g_l(B^*(z_0, \rho)) \quad \forall l \geq N,$$

see e.g. Theorem 46.I.1 in [15]. Consequently, $y_0 \in \text{Kern } D'_l$, i.e. $D' \subset \text{Kern } D'_l$ by arbitrariness of y_0 . \square

Remark 1. In particular, Proposition 2 implies that $D' := g(D) \subset \mathbb{R}^n$ if $D'_l := g_l(D) \subset \mathbb{R}^n$ for all $l = 1, 2, \dots$.

The following statement for the plane case can be found in the paper [12], see also Lemma 2.16 in the monograph [8].

Lemma 2. *Let D be a domain in $\overline{\mathbb{R}}^n$, $l \in \{1, 2, \dots\}$, and let f_l be a sequence of homeomorphisms from D into $\overline{\mathbb{R}}^n$ such that f_l converges as $l \rightarrow \infty$ locally uniformly with respect to the spherical metric to a homeomorphism f of D into $\overline{\mathbb{R}}^n$. Then $f_l^{-1} \rightarrow f^{-1}$ locally uniformly in $f(D)$, too.*

Proof. Given a compactum $C \subset f(D)$, we have by Proposition 2 that $C \subset f_l(D)$ for all $l \geq l_0 = l_0(C)$. Set $g_l = f_l^{-1}$ and $g = f^{-1}$. The locally uniform convergence $g_l \rightarrow g$ is equivalent to the so-called continuous convergence, meaning that $g_l(u_l) \rightarrow g(u_0)$ for every convergent sequence $u_l \rightarrow u_0$ in $f(D)$; see e.g. [5], p. 268 or Theorems 20.VIII.2 and 21.X.4 in [14]. So, let $u_l \in f(D)$, $l \in \{0, 1, 2, \dots\}$ and $u_l \rightarrow u_0$ as $l \rightarrow \infty$. Let us show that $z_l := g_l(u_l) \rightarrow z_0 := g(u_0)$ as $l \rightarrow \infty$.

It is known that every metric space is an \mathcal{L}^* -space, i.e. a space with a convergence (see, e.g., Theorem 21.II.1 in [14]), and the Urysohn axiom for compact spaces says that $z_l \rightarrow z_0$ as

$l \rightarrow \infty$ if and only if, for every convergent subsequence $z_{l_k} \rightarrow z_*$, the equality $z_* = z_0$ holds; see e.g. the definition 20.I.3 in [14]. Hence it suffices to prove that the equality $z_* = z_0$ holds for every convergent subsequence $z_{l_k} \rightarrow z_*$ as $k \rightarrow \infty$. Let D_0 be a subdomain of D such that $z_0 \in D_0$ and $\overline{D_0}$ is a compact subset of D . Then by Proposition 2, $f(D_0) \subset \text{Kern} f_l(D_0)$ and hence u_0 together with its neighborhood belongs to $f_{l_k}(D_0)$ for all $k \geq K$. Thus, with no loss of generality we may assume that $u_{l_k} \in f_{l_k}(D_0)$, i.e. $z_{l_k} \in D_0$ for all $k \in \{1, 2, \dots\}$, and, consequently, $z_* \in D$. Then, by the continuous convergence $f_l \rightarrow f$, we have that $f_{l_k}(z_{l_k}) \rightarrow f(z_*)$, i.e. $f_{l_k}(g_{l_k}(u_{l_k})) = u_{l_k} \rightarrow f(z_*)$. The latter condition implies that $u_0 = f(z_*)$, i.e. $f(z_0) = f(z_*)$ and hence $z_* = z_0$. \square

The following statement for the plane case can be found in the paper [26], see also Proposition 2.6 in the monograph [8].

Theorem 1. *Let D be a domain in \mathbb{R}^n , $n \geq 2$, and let f_m , $m \in \{1, 2, \dots\}$, be a sequence of homeomorphisms of D into \mathbb{R}^n converging locally uniformly to a discrete mapping $f: D \rightarrow \mathbb{R}^n$ with respect to the spherical metric. Then f is a homeomorphism of D into \mathbb{R}^n .*

Proof. First of all, let us show by contradiction that f is injective. Indeed, let us assume that there exist $x_1, x_2 \in D$, $x_1 \neq x_2$, with $f(x_1) = f(x_2)$ and that $x_1 \neq \infty$. Set $B_t = B(x_1, t)$. Let t_0 be such that $\overline{B_t} \subset D$ and $x_2 \notin \overline{B_t}$ for every $t \in (0, t_0]$. By the Jordan–Brouwer theorem, see e.g. Theorem 4.8.15 in [28], $f_m(\partial B_t) = \partial f_m(B_t)$ splits \mathbb{R}^n into two components $C_m := f_m(B_t)$, $C_m^* = \mathbb{R}^n \setminus \overline{C_m}$.

By construction $y_m := f_m(x_1) \in C_m$ and $z_m := f_m(x_2) \in C_m^*$. Remark that the ball $B^*(y_m, h(y_m, \partial C_m))$ is contained inside of C_m and, consequently, its closure is inside of $\overline{C_m}$. Hence

$$h(y_m, \partial C_m) < h(y_m, z_m), \quad m \in \{1, 2, \dots\}. \quad (3)$$

By compactness of $\partial C_m = f_m(\partial B_t)$, there is $x_{m,t} \in \partial B_t$ such that

$$h(y_m, \partial C_m) = h(y_m, f_m(x_{m,t})), \quad m \in \{1, 2, \dots\}. \quad (4)$$

By compactness of ∂B_t , for every $t \in (0, t_0]$, there is $x_t \in \partial B_t$ such that $h(x_{m_k,t}, x_t) \rightarrow 0$ as $k \rightarrow \infty$ for some subsequence m_k . Since the locally uniform convergence of continuous functions in a metric space implies the continuous convergence (see [5], p. 268 or Theorem 21.X.3 in [14]), we have that $h(f_{m_k}(x_{m_k,t}), f(x_t)) \rightarrow 0$ as $k \rightarrow \infty$. Consequently, from (3) and (4) we obtain that $h(f(x_1), f(x_t)) \leq h(f(x_1), f(x_2)) \forall t \in (0, t_0]$. However, by the above assumption $f(x_1) = f(x_2)$ and we have $f(x_t) = f(x_1)$ for every $t \in (0, t_0]$. The latter condition contradicts the discreteness of f . Thus, f is injective.

It remains to show that f and f^{-1} are continuous. The mapping f is continuous as a locally uniform limit of continuous mappings, see e.g. Theorem 13.VI.3 in [14]. Finally, f^{-1} is continuous by Corollary 2. \square

4. Convergence of homeomorphisms and moduli. Later on, the following lemma plays a very important role. Its plane analog can be found in the paper [4], see also supplement A1 in the monograph [8].

Lemma 3. *Let f_m , $m \in \{1, 2, \dots\}$, be a sequence of homeomorphisms of a domain $D \subseteq \mathbb{R}^n$ into \mathbb{R}^n , $n \geq 2$, converging to a mapping f uniformly on every compact set in D with respect to the spherical metric in \mathbb{R}^n . Suppose that for every $x_0 \in D$ there exist sequences $R_k > 0$ and $r_k \in (0, R_k)$, $k \in \{1, 2, \dots\}$, such that $R_k \rightarrow 0$ as $k \rightarrow \infty$ and $\text{mod } f_m(A(x_0, r_k, R_k)) \rightarrow \infty$*

as $k \rightarrow \infty$ uniformly with respect to $m \in \{1, 2, \dots\}$. Then the mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism of D into \mathbb{R}^n .

Proof. Assume that f is not constant. Let us consider the open set V consisting of all points in D which have neighborhoods where f is a constant and show that $f(x) \neq f(x_0)$ for every $x_0 \in D \setminus V$ and $x \neq x_0$. Without loss of generality, we may assume that $f(x_0) \neq \infty$. Now, let us fix a point $x_* \neq x_0$ in $D \setminus V$ and choose $k \in \{1, 2, \dots\}$ such that $R := R_k < |x_* - x_0|$ and

$$\text{mod } f_m(A(x_0, r, R)) > (\omega_{n-1}/\tau_n(1))^{1/(n-1)}, \quad \forall m \in \{1, 2, \dots\} \quad (5)$$

for $r = r_k$ where $\tau_n(s)$ denotes the capacity of the Teichmüller ring $R_{T,n}(s) := [\mathbb{R}^n \setminus \{te_1 : t \geq s\}, [-e_1, 0]]$, $s \in (0, \infty)$.

Let $c_m \in f_m(S(x_0, R))$ and $b_m \in f_m(S(x_0, r))$ be such that

$$\min_{w \in f_m(S(x_0, R))} |w - f_m(x_0)| = |c_m - f_m(x_0)|, \quad \max_{w \in f_m(S(x_0, r))} |w - f_m(x_0)| = |b_m - f_m(x_0)|.$$

Since f_m is a homeomorphism, the set $f_m(A(x_0, r, R))$ is a ring domain $\mathfrak{R}_m = (C_m^1, C_m^2)$, where $a_m := f_m(x_0)$ and $b_m \in C_m^1$, c_m and $\infty \in C_m^2$. Applying Lemma 7.34 in [30] with $a = a_m$, $b = b_m$ and $c = c_m$, we obtain that

$$\text{cap } \mathfrak{R}_m = M(\Gamma(C_m^1, C_m^2)) \geq \tau_n \left(\frac{|a_m - c_m|}{|a_m - b_m|} \right). \quad (6)$$

Note that the function $\tau_n(s)$ is strictly decreasing (see Lemma 7.20 in [30]). Thus, it follows from (5) and (6) that

$$\frac{|a_m - c_m|}{|a_m - b_m|} \geq \tau_n^{-1}(\text{cap } \mathfrak{R}_m) > \tau_n^{-1}(\tau_n(1)) = 1.$$

Hence there is a spherical ring $A_m = \{y \in \mathbb{R}^n : \rho_m < |y - f_m(x_0)| < \rho_m^*\}$ in the ring domain \mathfrak{R}_m for every $m \in \{1, 2, \dots\}$. Since f is not locally constant at x_0 , we can find a point x' in the ball $|x - x_0| < r$ with $f(x_0) \neq f(x')$. The ring A_m separates $f_m(x_0)$ and $f_m(x')$ from $f_m(x_*)$ and, thus, $|f_m(x') - f_m(x_0)| \leq \rho_m$ and $|f_m(x_*) - f_m(x_0)| \geq \rho_m^*$. Consequently, $|f_m(x') - f_m(x_0)| \leq |f_m(x_*) - f_m(x_0)|$ for all $m \in \{1, 2, \dots\}$. Under $m \rightarrow \infty$ we have then $0 < |f(x') - f(x_0)| \leq |f(x_*) - f(x_0)|$ and hence $f(x_*) \neq f(x_0)$.

It remains to show that the set V is empty. Let us assume that V has a nonempty component V_0 . Then $f(x) \equiv z$ for every $x \in V_0$ and some $z \in \overline{\mathbb{R}}^n$. Note that $\partial V_0 \cap D \neq \emptyset$ by connectedness of D , because $f \not\equiv \text{const}$ in D and the set $D \setminus \overline{V_0}$ is also open. If $x_0 \in \partial V_0 \cap D$, then by continuity, $f(x_0) = z$ contradicting the assertion established in the first part of the proof because $x_0 \in D \setminus V$.

Thus, we have proved that the mapping f is injective if f is not constant. But f is continuous as a locally uniform limit of continuous mappings f_m , see Theorem 13.VI.3 in [14], and then by Corollary 2 f is a homeomorphism. Finally, by Remark 1 $f(D) \subset \mathbb{R}^n$ and the proof is complete. \square

Lemma 4. Let D be a domain in \mathbb{R}^n , $n \geq 2$, $Q_m : D \rightarrow (0, \infty)$ be measurable functions, f_m , $m \in \{1, 2, \dots\}$, be a sequence of ring Q_m -homeomorphisms of D into \mathbb{R}^n converging locally uniformly to a mapping f . Suppose

$$\int_{\varepsilon < |x - x_0| < \varepsilon_0} Q_m(x) \cdot \psi^n(|x - x_0|) dm(x) = o(I^n(\varepsilon, \varepsilon_0)) \quad \forall x_0 \in D, \quad (7)$$

where $o(I^n(\varepsilon, \varepsilon_0))/I^n(\varepsilon, \varepsilon_0) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly with respect to m for $\varepsilon_0 < \text{dist}(x_0, \partial D)$ and a measurable function $\psi(t): (0, \varepsilon_0) \rightarrow [0, \infty]$ such that

$$0 < I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (8)$$

Then the mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism into \mathbb{R}^n .

Remark 2. In particular, the conclusion of Lemma 4 holds for Q -homeomorphisms f_m with a measurable function $Q: D \rightarrow (0, \infty)$ such that

$$\int_{\varepsilon < |x - x_0| < \varepsilon_0} Q(x) \cdot \psi^n(|x - x_0|) dm(x) = o(I^n(\varepsilon, \varepsilon_0)) \quad \forall x_0 \in D. \quad (9)$$

Proof. By Luzin's theorem, there exists a Borel function $\psi_*(t)$ such that $\psi(t) = \psi_*(t)$ for a.e. $t \in (0, \varepsilon_0)$, see e.g. 2.3.6 in [6]. Since $Q_m(x) > 0$ for all $x \in D$ we have from (7) that $I(\varepsilon, a) \rightarrow \infty$ for every fixed $a \in (0, \varepsilon_0)$ and, in particular, $I(\varepsilon, a) > 0$ for every $\varepsilon \in (0, b)$ and some $b = b(a) \in (0, a)$. Given $x_0 \in D$ and a sequence of such numbers $b = \varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, $k \in \{1, 2, \dots\}$, consider the sequence of the Borel measurable functions $\rho_{\varepsilon, k}$ defined by

$$\rho_{\varepsilon, k}(x) = \begin{cases} \psi_*(|x - x_0|)/I(\varepsilon, \varepsilon_k), & \varepsilon < |x - x_0| < \varepsilon_k, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the function $\rho_{\varepsilon, k}(x)$ is admissible for $\Gamma_{\varepsilon, k} := \Gamma(S(x_0, \varepsilon), S(x_0, \varepsilon_k), A(x_0, \varepsilon, \varepsilon_k))$ because

$$\int_{\gamma} \rho_{\varepsilon, k}(x) |dx| \geq \frac{1}{I(\varepsilon, \varepsilon_k)} \int_{\varepsilon}^{\varepsilon_k} \psi(t) dt = 1$$

for all (locally rectifiable) curves $\gamma \in \Gamma_{\varepsilon, k}$ (see Theorem 5.7 in [29]). Then by the definition of ring Q -homeomorphisms

$$M(f_m(\Gamma_{\varepsilon, k})) \leq \frac{1}{I^n(\varepsilon, \varepsilon_k)} \int_{\varepsilon < |x - x_0| < \varepsilon_0} Q(x) \cdot \psi^n(|x - x_0|) dm(x) \quad (10)$$

for all $m \in \mathbb{N}$. Note that $\frac{1}{I^n(\varepsilon, \varepsilon_k)} = \alpha_{\varepsilon, k} \cdot \frac{1}{I^n(\varepsilon, \varepsilon_0)}$, where $\alpha_{\varepsilon, k} := \left(1 + \frac{I(\varepsilon_k, \varepsilon_0)}{I(\varepsilon, \varepsilon_k)}\right)^n$ is independent on m and bounded as $\varepsilon \rightarrow 0$. Then it follows from (7) and (10) that there exists $\varepsilon_k^* \in (0, \varepsilon_k)$ such that for all $M(f_m(\Gamma_{\varepsilon_k^*, k})) \leq 2^{-k} \quad \forall m \in \mathbb{N}$. Applying Lemma 3 we obtain the desired conclusion. \square

The next important statements follows from Lemma 4.

Theorem 2. Let D be a domain in \mathbb{R}^n , $n \geq 2$, $Q: D \rightarrow (0, \infty)$ a Lebesgue measurable function and let f_m , $m \in \{1, 2, \dots\}$, be a sequence of ring Q -homeomorphisms of D into \mathbb{R}^n converging locally uniformly to a mapping f . Suppose that $Q \in \text{FMO}$. Then the mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism into \mathbb{R}^n .

Proof. Let $x_0 \in D$. We may consider further that $x_0 = 0 \in D$. Choosing a positive $\varepsilon_0 < \min\{\text{dist}(0, \partial D), e^{-1}\}$, we obtain by Lemma 1 for the function $\psi(t) = \frac{1}{t \log \frac{1}{t}}$ that

$$\int_{\varepsilon < |x| < \varepsilon_0} Q(x) \cdot \psi^n(|x|) dm(x) = O\left(\log \log \frac{1}{\varepsilon}\right).$$

Note that $I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t) dt = \log \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon_0}}$. Now the desired conclusion follows from Lemma 4. \square

The following conclusions can be obtained on the basis of Theorem 2, Proposition 1 and Corollary 1.

Corollary 3. *In particular, the limit mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism of D into \mathbb{R}^n whenever*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0, \varepsilon)} Q(x) dm(x) < \infty \quad \forall x_0 \in D$$

or whenever every $x_0 \in D$ is a Lebesgue point of Q .

Theorem 3. *Let D be a domain in \mathbb{R}^n , $n \geq 2$, and let $Q: D \rightarrow (0, \infty)$ be a measurable function such that*

$$\int_0^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} = \infty \quad \forall x_0 \in D \quad (11)$$

for a positive $\varepsilon(x_0) < \text{dist}(x_0, \partial D)$ where $q_{x_0}(r)$ denotes the average of $Q(x)$ over the sphere $|x - x_0| = r$. Suppose that f_m , $m \in \{1, 2, \dots\}$, is a sequence of ring Q -homeomorphisms from D into \mathbb{R}^n converging locally uniformly to a mapping f . Then the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into \mathbb{R}^n .

Proof. Fix $x_0 \in D$ and set $I = I(\varepsilon, \varepsilon_0) = \int_{\varepsilon}^{\varepsilon_0} \psi(t) dt$, $\varepsilon \in (0, \varepsilon_0)$, where

$$\psi(t) = \begin{cases} 1/[t q_{x_0}^{\frac{1}{n-1}}(t)], & t \in (\varepsilon, \varepsilon_0), \\ 0, & t \notin (\varepsilon, \varepsilon_0). \end{cases}$$

Note that $I(\varepsilon, \varepsilon_0) < \infty$ for every $\varepsilon \in (0, \varepsilon_0)$. Indeed, by Theorem 3.15 in [20] on the criterion of ring Q -homeomorphisms, we have that

$$M\left(f\left(\Gamma(S(x_0, \varepsilon), S(x_0, \varepsilon_0), A(x_0, \varepsilon, \varepsilon_0))\right)\right) \leq \frac{\omega_{n-1}}{I^{n-1}}. \quad (12)$$

On the other hand, by Lemma 1.15 in [18], we see that

$$M\left(\Gamma(f(S(x_0, \varepsilon)), f(S(x_0, \varepsilon_0)), f(A(x_0, \varepsilon, \varepsilon_0)))\right) > 0.$$

Then it follows from (12) that $I < \infty$ for every $\varepsilon \in (0, \varepsilon_0)$. In view of (11), we obtain that $I(\varepsilon, \varepsilon_*) > 0$ for all $\varepsilon \in (0, \varepsilon_*)$ with some $\varepsilon_* \in (0, \varepsilon_0)$. Finally, simple calculations show that (9) holds, in fact,

$$\int_{\varepsilon < |x - x_0| < \varepsilon_*} Q(x) \cdot \psi^n(|x - x_0|) dm(x) = \omega_{n-1} \cdot I(\varepsilon, \varepsilon_*)$$

and $I(\varepsilon, \varepsilon_*) = o(I^n(\varepsilon, \varepsilon_*))$ by (11). The rest follows by Lemma 4. \square

Corollary 4. *In particular, the conclusion of Theorem 3 holds if $q_{x_0}(r) = O(\log^{n-1} \frac{1}{r})$ for all $x_0 \in D$.*

Corollary 5. *Under assumptions of Theorem 3, the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into \mathbb{R}^n provided $Q(x)$ has singularities only of the logarithmic type of the order which is not more than $n - 1$ at every point $x_0 \in D$.*

Theorem 4. Let D be a domain in \mathbb{R}^n , $n \geq 2$, and $Q: D \rightarrow (0, \infty)$ be a measurable function such that

$$\int_{\varepsilon < |x-x_0| < \varepsilon_0} \frac{Q(x)}{|x-x_0|^n} dm(x) = o\left(\log^n \frac{1}{\varepsilon}\right) \quad \forall x_0 \in D \quad (13)$$

as $\varepsilon \rightarrow 0$ for some positive number $\varepsilon_0 = \varepsilon(x_0) < \text{dist}(x_0, \partial D)$. Suppose that f_m , $m \in \{1, 2, \dots\}$, is a sequence of ring Q -homeomorphisms from D into $\overline{\mathbb{R}^n}$ converging locally uniformly to a mapping f . Then the limit mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into \mathbb{R}^n .

Proof. The conclusion follows from Lemma 4 by the choice $\psi(t) = \frac{1}{t}$. \square

For every nondecreasing function $\Phi: [0, \infty] \rightarrow [0, \infty]$, the inverse function $\Phi^{-1}: [0, \infty] \rightarrow [0, \infty]$ can be well defined by setting $\Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t$. As usual, here \inf is equal to ∞ if the set of all $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is nondecreasing, too. Note also that if $h: [0, \infty] \rightarrow [0, \infty]$ is a sense-preserving homeomorphism and $\varphi: [0, \infty] \rightarrow [0, \infty]$ is a nondecreasing function, then

$$(\varphi \circ h)^{-1} = h^{-1} \circ \varphi^{-1}. \quad (14)$$

Theorem 5. Let D be a domain in \mathbb{R}^n , $n \geq 2$, let $Q: D \rightarrow (0, \infty)$ be a measurable function and $\Phi: [0, \infty] \rightarrow [0, \infty]$ be a nondecreasing convex function. Suppose that

$$\int_D \Phi(Q(x)) \frac{dm(x)}{(1+|x|^2)^n} \leq M < \infty \quad (15)$$

and

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty \quad (16)$$

for some $\delta > \Phi(0)$. Suppose that f_m , $m \in \{1, 2, \dots\}$, is a sequence of ring Q -homeomorphisms of D into \mathbb{R}^n converging locally uniformly to a mapping f . Then the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into \mathbb{R}^n .

Proof. It follows from (15)–(16) and Theorem 3.1 in [21] that the integral in (11) is divergent for some positive $\varepsilon(x_0) < \text{dist}(x_0, \partial D)$. The rest follows by Theorem 3. \square

Remark 3. We may assume in Theorem 5 that the function $\Phi(t)$ is convex not on the whole segment $[0, \infty]$ but only on the segment $[t_*, \infty]$ where $t_* = \Phi^{-1}(\delta)$. Indeed, every non-decreasing function $\Phi: [0, \infty] \rightarrow [0, \infty]$ which is convex on the segment $[t_*, \infty]$ can be replaced with a non-decreasing convex function $\Phi_*: [0, \infty] \rightarrow [0, \infty]$ in the following way. Set $\Phi_*(t) \equiv 0$ for $t \in [0, t_*]$, $\Phi(t) = \varphi(t)$ for $t \in [t_*, T_*]$ and $\Phi_* \equiv \Phi(t)$ for $t \in [T_*, \infty]$, where $\tau = \varphi(t)$ is the line passing through the point $(0, t_*)$ and touching the graph of the function $\tau = \Phi(t)$ at a point $(T_*, \Phi(T_*))$, $T_* \in (t_*, \infty)$. By the construction, we have that $\Phi_*(t) \leq \Phi(t)$ for all $t \in [0, \infty]$ and $\Phi_*(t) = \Phi(t)$ for all $t \geq T_*$ and, consequently, conditions (15) and (16) hold for Φ_* under the same M and every $\delta > 0$.

Furthermore, by the same reasons it is sufficient to assume that the function Φ is only minorized by a nondecreasing convex function Ψ on a segment $[T, \infty]$ such that

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau [\Psi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty \quad (17)$$

for some $T \in [0, \infty)$ and $\delta > \Psi(T)$. Note that condition (17) can be written in terms of the function $\psi(t) = \log \Psi(t)$

$$\int_{\Delta}^{\infty} \psi(t) \frac{dt}{t^{n'}} = \infty \quad (18)$$

for some $\Delta > t_0 \in [T, \infty]$, where $t_0 := \sup_{\psi(t)=-\infty} t$, $t_0 = T$ if $\psi(T) > -\infty$, and where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is decreasing in n and $n' = n/(n-1) \rightarrow 1$ as $n \rightarrow \infty$, see Proposition 2.3 in [21]. It is clear that if the function ψ is nondecreasing and convex, then the function $\Phi = e^{\psi}$ is so but the inverse conclusion generally speaking is not true. However, the conclusion of Theorem 5 is valid if $\psi^m(t)$, $t \in [T, \infty]$, is convex and (18) holds for ψ^m under some $m \in \mathbb{N}$ because $e^{\tau} \geq \tau^m/m!$ for all $m \in \mathbb{N}$.

Corollary 6. *In particular, the conclusion of Theorem 5 is valid if, for some $\alpha > 0$,*

$$\int_D e^{\alpha Q^{\frac{1}{n-1}}(x)} \frac{dm(x)}{(1+|x|^2)^n} \leq M < \infty.$$

The same is true for any function $\Phi = e^{\psi}$, where $\psi(t)$ is a finite product of the function αt^{β} , $\alpha > 0$, $\beta \geq 1/(n-1)$, and some of the functions $[\log(A_1 + t)]^{\alpha_1}$, $[\log \log(A_2 + t)]^{\alpha_2}, \dots$, $\alpha_m \geq -1$, $A_m \in \mathbb{R}$, $m \in \mathbb{N}$, $t \in [T, \infty]$, $\psi(t) \equiv \psi(T)$, $t \in [0, T]$.

Remark 4. For further applications, integral conditions (15) and (16) for Q and Φ can be written in other forms that are more convenient for some cases. Namely, by (14) with $h(t) = t^{\frac{1}{n-1}}$ and $\varphi(t) = \Phi(t^{n-1})$, $\Phi = \varphi \circ h$, the couple of conditions (15) and (16) is equivalent to the following couple

$$\int_D \varphi \left(Q^{\frac{1}{n-1}}(x) \right) \frac{dm(x)}{(1+|x|^2)^n} \leq M < \infty \quad (19)$$

and

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \varphi^{-1}(\tau)} = \infty \quad (20)$$

for some $\delta > \varphi(0)$. Moreover, by Theorem 2.1 in [27] the couple of the conditions (19) and (20) is in turn equivalent to the next couple

$$\int_D e^{\psi \left(Q^{\frac{1}{n-1}}(x) \right)} \frac{dm(x)}{(1+|x|^2)^n} \leq M < \infty \quad \text{and} \quad \int_{\Delta}^{\infty} \psi(t) \frac{dt}{t^2} = \infty$$

for some $\Delta > t_0$, where $t_0 := \sup_{\psi(t)=-\infty} t$, $t_0 = 0$ if $\psi(0) > -\infty$.

Finally, as it follows from Lemma 4 all the results of this section are valid if f_m are Q_m -homeomorphisms and the above conditions on Q hold for Q_m uniformly with respect to the parameter $m \in \{1, 2, \dots\}$.

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